MA 441, Fall 2014

Assignment 5.

This homework is due *Thursday*, October 2.

There are total 32 points in this assignment. 25 points is considered 100%. If you go over 25 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.1–3.2 in Bartle–Sherbert.

(1) REMINDER. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* ε_0 that demonstrates that (x_n) does not converge to x.

  (a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

  (b) [2pt] $x_n = 1/n$, $x = 1/2014$.

(2) REMINDER. Recall definition of a sequence in $\mathbb{R}$ converging to an $x \in \mathbb{R}$: Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

Below you can find (erroneous!) "definitions" of a sequence converging to x. In each case describe, exactly which sequences are "converging to x" according to that "definition".

  (a) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \forall K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.
  (*If you are confused at this point, think of the problem this way: suppose for some sequence* (x_n) *and a number* $x \in \mathbb{R}$ *you know that statement (a) is true. What can you say about about* (x_n)*?*)

  (b) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\exists K \in \mathbb{N}\ \forall \varepsilon > 0\ \forall n > K, |x - x_n| < \varepsilon$.

  (c) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\exists \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

  (d) [4pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \exists n > K, |x - x_n| < \varepsilon$.

(3) Find limits using Squeeze Theorem:

  (a) [2pt] $\lim\limits_{n\to\infty} \frac{n^2+2014n(\sin n+3\cos n^7)-1}{2n^2-\cos(3n^2+1)}$,

  (b) [2pt] $\lim\limits_{n\to\infty} \sqrt{n^2 + \cos(2014n+1)} - \sqrt{n^2 - \sin(n^3 - 1)}$

(4) (a) [3pt] (Example 3.1.11e) Prove that $n^{1/n} \to 1$ $(n \to \infty)$.

  (b) [2pt] (3.2.14a) Use Squeeze theorem to find limit of the sequence (n^{1/n^2}).

(5) [3pt] (Theorem 3.1.9) Let X be a sequence in $\mathbb{R}$ and X_m its m-tail, $m \in \mathbb{N}$. Prove that X converges to some $L \in \mathbb{R}$ if and only if X_m converges to the same L.

(6) [3pt] (3.2.8) Find a mistake in the following argument.
"Find $\lim_{n\to\infty}(1+\frac{1}{n})^n$ as shown below:

$$\begin{aligned}\lim_{n\to\infty}(1+\tfrac{1}{n})^n &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n})\\ &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot\lim_{n\to\infty}(1+\tfrac{1}{n})\cdots\lim_{n\to\infty}(1+\tfrac{1}{n})\\ &= \Big(\lim_{n\to\infty}(1+\tfrac{1}{n})\Big)^n = 1^n = 1.\end{aligned}$$

Therefore, $\lim_{n\to\infty}(1+\frac{1}{n})^n = 1$."
Comment. To reiterate, the argument above is erroneous and the obtained value of the limit is wrong, too.